COMPLEXES OF RINGS

BY

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ABSTRACT

Homology group of complexes of finitely generated projective modules are shown to be torsion groups, and a simplified proof of the vanishing of the cohomology groups $n \geqq 3$ of inseparable extensions is given.

This paper contains results on two different topics concerning complexes of rings: 1) Homology groups for complexes of finitely generated projective R-algebras, and 2) Cohomology groups of inseparable extensions.

The complexes of fields which have been introduced in [1], were extended by Rosenberg and Zelinsky to arbitrary commutative R-algebras, and their cohomology groups have been studied. In [2], homology groups have been introduced using the notation of a norm but this could be applied only for free algebras and, in particular, for field extensions. Recently, O. Goldman [4] has given a satisfactory definition for determinant of endomorphisms of finitely generated projective R-modules (R—a commutative ring). This seems to be the right background for defining a norm for arbitrary finitely generated projectives R-algebra S, and after proving the basic properties of this norm—the result on the homology group of [2] are generalized to this case. As a consequence it is shown that the cohomology group are of torsion groups with exponent depending on the maximum of the p-ranks of S, which are the dimension of the spaces $S \otimes R_p$ where R_p is the local ring of quotients of R with respect to the prime ideals p of R.

The second part contains a different proof of a result of Berkson [3] that $H^n(F/C) = 0$, $n \geqq 3$ for inseparable extensions F of C of exponent 1. The proof is simpler and probably can be carried over to a more general case but no attempt has been made here.

1. **Determinants and Norms.** Let E be a finitely generated (f.g) projective R-module, and $\alpha \in \mathrm{Hom}_R(E, E)$. If E is free then the determinant $\det \alpha$ is a well-defined element of R, and in the general case we adopt Goldman's defintion ([4]) of the determinant which is obtained as follows:

Let $E \oplus E_1 = F$, and F a f.g. free R-module. If e_1 denotes the identity transformation of E_1 and $\alpha_1 = \alpha \oplus e_1$, then set $\det \alpha = \det \alpha_1$ where the latter is defined

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in the classical way. It is shown in [4] that det α is independent on E_1 or F and it has most of the properties of the determinant.

Let $R[x]$ be the ring of polynomials in one indeterminate over R, then the characteristic polynomial of α is defined as $\det(x - \alpha \otimes 1) = \phi(\alpha, E, x)$, where $\alpha \otimes 1$ is the $R[x]$ endomorphism of $E \otimes R[x]$.

If p is a prime ideal in R, and R_p is the local ring of R at p, then $E \otimes R_p$ is free of finite rank—which is called the p-rank of E. We shall use the following result on the characteristic polynomial of the zero ([4, Proposition 2.2, Theorem 3.1]):

$$(1.1) \qquad \phi(0, E, x) = \sum_{i=0}^{n} e_i x^i,$$

where the e_j are mutual orthogonal idempotents and $1 = \Sigma e_i$. For every prime ideal p, there exists exactly one $e_j \notin p$, and then the p-rank of E is j; and for each $e_j \neq 0$ there exists a p such the p-rank of E is j.

We shall need the following additional properties of the determinants:

PROPOSITION 1: (a) $\phi(\alpha, E, 0) = \det(-\alpha)$.

(b) For every $\lambda \in R$, $\phi(\alpha - \lambda, E, x) = \phi(\alpha, E, x + \lambda)$.

(c) If $\lambda \in F$ then $\det \lambda = \Sigma_{i=0}^{n} e_i \lambda^i$.

Proof. The basic tool in the proof is the application of [4. Proposition 1.4] which states:

(1.2) If $f: R \to S$ is a ring homomorphism, and $\alpha \otimes 1 \in \mathrm{Hom}_S(E \otimes S, E \otimes S)$, where S is given the structure of an R-module by f, then $\det(\alpha \otimes 1) = f(\det \alpha)$.

Consider the homomorphism $f: R[x] \to R$ given by setting $f[\phi(x)] = \phi(0)$, and then $(E \otimes R[x]) \otimes R$ is identified with E, then $\det(-\alpha) = \det[(x - \alpha \otimes 1) \otimes 1] = = f \cdot \det(x - \alpha \otimes 1) = \phi(\alpha, E, 0)$ which proves (a).

The proof of (b) is obtained similarly by considering the homomorphism $f_1: R[x] \to R[x]$, by setting $f[\phi(x)] = \phi(x + \lambda)$. Here $(E \otimes R[x]) \otimes R[x] \cong E \otimes R[x]$ by identifying $v \otimes 1 \otimes \phi[x]$ with $v \otimes \phi[x]$ and $[x - \alpha \otimes 1] \otimes 1$ is identified with $x + \lambda - \alpha \otimes 1$, since $[(x - \alpha \otimes 1) \otimes 1](v \otimes 1 \otimes 1) = v \otimes x \otimes 1 - \alpha v \otimes 1 \otimes 1$ $= v \otimes 1 \otimes fx - \alpha v \otimes 1 \otimes 1 = v \otimes 1 \otimes (x + \lambda) - \alpha v \otimes 1 \otimes 1 = v \otimes 1 \otimes x$ $- (\alpha - \lambda) v \otimes 1 \otimes x$ so that applying (1.2) we obtain

$$\det(x - (\alpha - \lambda) \otimes 1) = f \det(x - \alpha \otimes 1) = \phi(\alpha, E, x + \lambda)$$

which proves (b).

The last result is a simple consequence of (a) and (b) obtained by setting $\alpha = 0$, $x = 0$ and using (1.1).

Next we turn to the notion of the Norm, and we consider henceforth two commutative rings $S \supseteq R$ with the same unit, and such that S is f. g. R-module and

R projective. Let $a \in S$ and $a_r: u \to ua$ be the R-endomorphism of S obtained by multiplication by a. We define

DEFINITION. $\operatorname{Norm}(S/R; a) = \det a_r$.

The following properties of the Norm will be used:

PROPOSITION 2. (a) Let S be an R-algebra, S' and R'-algebra and $\sigma: S \to S'$ a ring isomorphism which maps R on R' then $\sigma \operatorname{Norm}(S/R; a) = \operatorname{Norm}(S'/R'; \sigma a)$.

(b) Let E be a f.g. S-projective module, and $\alpha \in \operatorname{Hom}_S(E, E)$. Consider E also as an R-module, then E is f.g. and R-projective. If $\det_S \alpha$, $\det_R \alpha$ denote the determinants of α considered as an element of $\operatorname{Hom}_S(E, E)$ and $\operatorname{Hom}_R(E, E)$ respectively, then

$$\det_R \alpha = \operatorname{Norm}(S/R, \det_S \alpha).$$

Proof. Let R' be given the structure of an R-algebra by σ, i.e., $r \cdot r' = \sigma(r) r'$, $r \in R$ and $r' \in R'$. Then $S \otimes_R R'$ can be identified with S' by setting $s \otimes r' = \sigma(s) r'$ and with this identification $a_r \otimes 1 = \sigma(a)_r$. Hence, it follows by (1.2) that
$\operatorname{Norm}(S'/R'; \sigma(a)) = \det[\sigma(a)_r] = \det(a_r \otimes 1) = \sigma \det a_r = \sigma \operatorname{Norm}(S/R; a)$(1)

If E is S-free and S is R-free then part (b) is well known ([6, Proposition 7, p. 140]). The general case will be obtained by reduction to the free case:

First we note that E is also a f.g. projective R-module. Indeed, clearly it is f.g. over R; and if $E \oplus E' = \sum S u_i$ is free S-module, then since $S u_i \cong S$, and S is R-projective it follows that $S u_i \oplus S_i$ is R-free for some R-module S_i, and consequently $E \oplus E' \oplus \sum S_i = \sum (S u_i \oplus S_i)$ is R-free which proves that E is R-projective.

Next we observe that it suffices to consider only free S-modules. For, assume that (b) holds for free modules, and E be an arbitrary projective module, then $E \oplus E' = F$ and F a f.g. and free over S, for some E'. Then, $\alpha_1 = \alpha \oplus e_1$ and $\det_S \alpha = \det_S \alpha_1$ by definition.

It follows now by [4] proposition 1.5, that $\det_R \alpha_1 = \det_R \alpha \det_R e_1$, and since e_1 is the identity $\det_R e_1 = 1$ so that $\det_R \alpha_1 = \det_R \alpha$. Hence

$$\det_R \alpha = \det_R \alpha_1 = \operatorname{Norm}(S/R; \det_S \alpha_1) = \operatorname{Norm}(S/R; \det_S \alpha)$$

since (b) was assumed to be valid for α_1.

Consider now the element $r = \det_R \alpha - \operatorname{Norm}(S/R; \det_S \alpha)$. Let $\mathcal{M}$ be a maximal idea in R, and $R_{\mathcal{M}}$ be the local ring at $\mathcal{M}$. Thus, $S_{\mathcal{M}} = S \otimes R_{\mathcal{M}}$ is $R_{\mathcal{M}}$-free and $E_{\mathcal{M}} = E \otimes R_{\mathcal{M}}$ can be considered as an $S_{\mathcal{M}}$-module, and assuming that E is S-free then $E_{\mathcal{M}}$ is also $S_{\mathcal{M}}$ free.

(1) This simplified proof is due to the referee.

Let $f: R \to R_{\mathcal{M}}$ and its extension $\bar{f}: S \to S \otimes R_{\mathcal{M}}$. It follows by (1.2) that $\det_{R\mathcal{M}}(\alpha \otimes 1) = f(\det_R \alpha)$ and $\det_S(\alpha \otimes 1) = \bar{f}(\det_S \alpha)$, where $\alpha \otimes 1$ in both cases is taken as an endomorphism of $E_{\mathcal{M}}$. Now $E_{\mathcal{M}}$, $S_{\mathcal{M}}$ are free over $R_{\mathcal{M}}$, hence:

$$\mathrm{Norm}\,[S_{\mathcal{M}}/R_{\mathcal{M}}; \det_{S_{\mathcal{M}}}(\alpha \otimes 1)] = \det_{R_{\mathcal{M}}}(\alpha \otimes 1).$$

Consequently:

$$\begin{aligned} f(r) &= f(\det_R \alpha) - f\,\mathrm{Norm}\,(S/R; \det_S \alpha) = \det_{R_{\mathcal{M}}}(\alpha \otimes 1) \\ &= f(\det\,[(\det_S \alpha)_r \otimes 1]) = \det_{R_{\mathcal{M}}}(\alpha \otimes 1) - f\,[\det\,[(\det_S \alpha \otimes 1)_r]] \\ &= \det_{R_{\mathcal{M}}}(\alpha \otimes 1) - \mathrm{Norm}\,[S_{\mathcal{M}}/R_{\mathcal{M}}\,; f(\det_S \alpha)] \\ &= \det_{R_{\mathcal{M}}}(\alpha \otimes 1) - \mathrm{Norm}\,(S_{\mathcal{M}}/R_{\mathcal{M}}\,; \det_{S\mathcal{M}} \alpha) = 0 \end{aligned}$$

This being true for every maximal ideal $\mathcal{M}$, yields by [4, lemma 1] that $\mathcal{M} = 0$, which completes the proof of (b).

A simple corollary of (b) is the transitivity property of the Norm:

COROLLARY 3. *Let $T \supset S \supset R$ each f.g. and projective over the preceding ring then $Norm(T/R; a) = Norm\,[T/S; Norm(S/R, a)]$.*

2. **Homology of Rings.**. The complex $C^*(S/R)$ was defined in [2] for fields S which are extensions of R, with the aid of the Norm, and this can be extended to arbitrary f.g. R-projective rings S as follows:

Let $S^n = S \otimes \cdots \otimes S (n \cdots$ terms and $\otimes$ taken with respect to R), the homomorphism $\varepsilon_i: S^{n-1} \to S^n$ are defined by setting:

$$\varepsilon_i(a_1 \otimes \cdots \otimes a_{n-1}) = a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_{n-1}. \tag{2.1}$$

S^n is also a f.g. projective $\varepsilon_i S^{n-1}$-module and we set

$$v_i(a) = \varepsilon_i^{\,-1}\mathrm{Norm}\,(S^n/\varepsilon_i S^{n-1}; a), \qquad a \in (S^n)^* \tag{2.2}$$

where ( )* denotes the corresponding multiplicative group of the invertible elements.

The complex $C^*(S/R)$ is the sequence of groups,

$$R \leftarrow S^* \leftarrow (S^2)^* \leftarrow \cdots \leftarrow (S^n)^* \to (S^{n+1})^*$$

with the derivation $\mathcal{N} : (S^n)^* \to (S^{n-1})^*$ defined by:

$$\mathcal{N}(a) = [v_1(a)\, v_3(a) \cdots]\,[v_2(a)\, v_4 \cdots]^{-1}.$$

The last map $S^* \to R$ is $\mathcal{N} = \mathrm{Norm}\,(S/R); \cdot\,)$. The Mappings $\mathcal{N}$ are well defined since a is invertible and therefore, [4, Proposition 1.3] implies that $v_i(a)$ is invertible and so $v_i(a)^{-1}$ exists.

The following relations hold between the ν_i, ε_i:

LEMMA 4. (2.3) $\nu_i\nu_j = \nu_j\nu_{i+1}$ *for* $i \geqq j$

(2.4) $\varepsilon_i\nu_j = \nu_{j+1}\varepsilon_i$ *for* $i \leqq j$ *and*

$\varepsilon_i\nu_j = \nu_j\varepsilon_{i+1}$ *for* $i \geqq j$.

The proof of (2.3) follows similarly to the proof of (1.7) of [2] using the transitivity of the Norm which holds also in our case by Corollary 3.

To prove (2.4), we use the relation

$$\varepsilon_i\varepsilon_j = \varepsilon_{j+1}\varepsilon_i, \qquad i \leqq j \tag{2.5}$$

and (1.2) in the following situation:

Consider S^n as $\varepsilon_j S^{n-1}$ module, and first let $i \leqq j$, then it follows by (2.5) that ε_i induces a map of: $\varepsilon_j S^{n-1} \to \varepsilon_{j+1} S^n$. Apply now (1.2) with $f = \varepsilon_i$ and $E = S^n$ then we have $\det(\varepsilon_i a)_r = \varepsilon_i \det a_r$, for $a \in S^n$. Since $a \otimes 1$ in our case is readily seen to be $\varepsilon_i a$. Thus:

$$\det(\varepsilon_i a)_r = \operatorname{Norm}(S^{n+1}/\varepsilon_{j+1}S^n, \varepsilon_i a) = \varepsilon_{j+1}\nu_{j+1}\varepsilon_i(a)$$

$$\varepsilon_i \det a_r = \varepsilon_i \operatorname{Norm}(S^n/\varepsilon_j S^n, a) = \varepsilon_i\varepsilon_j\nu_j(a).$$

Hence by (2.5) we get: $\varepsilon_{j+1}\nu_{j+1}\varepsilon_i(a) = \varepsilon_i\varepsilon_j\nu_j(a) = \varepsilon_{j+1}\varepsilon_i\nu_j(a)$. Cancelling ε_{j+1} of both sides yields the first part of (2.4).

The second part follows similarly:

Let $i > j$, so that (2.5) yields $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_{i-1}$ by interchanging i with $j+1$ and j with i. Here $\varepsilon_i : \varepsilon_j S^{n-1} \to \varepsilon_j S^{n-1}$, and as before the relation $\det(\varepsilon_i a)_r = \varepsilon_i \det a_r$, $a \in S^n$ will yield:

$$\det(\varepsilon_i a)_r = \operatorname{Norm}(S^{n+1}/\varepsilon_j S^n; \varepsilon_i a) = \varepsilon_j\nu_j\varepsilon_i(a)$$

$$\varepsilon_i \det a_r = \varepsilon_i \operatorname{Norm}(S^n/\varepsilon_j S^n; a) = \varepsilon_i\varepsilon_j\nu_j(a).$$

Hence, $\varepsilon_j\nu_j\varepsilon_i(a) = \varepsilon_i\varepsilon_j\nu_j(a) = \varepsilon_j\varepsilon_{i-1}\nu_j(a)$ which yields by cancelling ε_j the second part of (2.4), by replacing $i-1$ by i.

The property (2.3) yields as in [2] p. 4, the fact that $C_*(S/R)$ is a complex, and thus the homology groups $H_n(S/R)$ are well defined for arbitrary f.g. R-projective modules S.

Next we show that the 'restriction' and 'transfer' work for the general case as well.

Let T be an algebra which is a f.g. projective R-module, then the restriction $\rho^* : H_n(S/R) \to H_n(S \otimes T/T)$ was defined as the map induced by the complex homomorphism $\rho : C_*(S/R) \to C_*(S \otimes T/T)$ where $\rho : S^n \to S^n \otimes T$ is given by $\rho^n(a) = a \otimes 1$, followed by the isomorphism of the complexes $C_*(S/R, \otimes T)$ and $C_*(S \otimes T/T)$, where the first consists of the groups $(S^n \otimes_R T)^*$ and the latter consists of the groups $[(S \otimes_R T)_T]^*$ with the obvious derivations. The proof of this result as given in [2] depends on a choice of a base of the corresponding modules and in our case it will be replaced by (1.2).

THEOREM 5A. *The mappings* $\sigma^n: S^n \otimes T \to (S \otimes T)_T^n$ *given by*

$$\sigma^n(s_1 \otimes \cdots \otimes s_n \otimes t) = (s_1 \otimes 1) \otimes_T \cdots \otimes_T (s_n \otimes t), \qquad s_i \in S,$$

$t \in T$ *yield a complex isomorphism* $\sigma: C(S/R \otimes T) \to C(S \otimes T/T)$ *and the mappings* $\rho^n: S^n \to S^n \otimes T$, *given by* $\rho^n(a) = a \otimes 1$ *yield a complex homomorphism* $\rho: C_*(S/R) \to C_*(S/R, \otimes T)$.

The first part is an immediate consequence of the fact that σ is an isomorphism which commutes with the ε_j, and part (a) of proposition 2 yields

$$\sigma[\text{Norm}(S^n \otimes T/S^{n-1} \otimes T, a)] = \text{Norm}((S \otimes T)^n/(S \otimes T)^{n-1}, \sigma a)$$

from which we readily verify that $\sigma v_i = v_i\sigma$. In the proof of the second part we apply (1.2) to the homomorphisms $\rho_i: \varepsilon_i S^{n-1} \to \varepsilon_i S^{n-1} \otimes T$ which replaces f and S being considered as an $\varepsilon_i S^{n-1}$-module. Thus we get:

$$\det \rho a = \det(a \otimes 1) = \rho \det a, \qquad a \in S^n$$

from which it follows that $\text{Norm}(S^n \otimes T/(\varepsilon_i S^{n-1} \otimes T); a) = \rho[\text{Norm}(S^n/\varepsilon_i S^{n-1}; a)]$, and since ε_i commutes with ρ, one readily verifies that ρ is a complex homomorphism.

The map $\tau: C(S/R, \otimes T) \to C(S/R)$(2) is defined by $\tau^n(a) = \text{Norm}(S^n \otimes T/S^n, a)$ for $a \in S^n \otimes T$, and it yields the *transfer* map $(\tau\sigma^{-1})^*: H(S \otimes T/T) \to H(S/T)$. This is also true in the general case as we show that:

THEOREM 5B. τ *is a complex homomorphism.*

Indeed τ commutes with ε_j; for apply (1.2) with $\varepsilon_j: S^n \to S^{n+1}$ and $E = S^n \otimes T$, so that $E \otimes S^{n+1} = S^{n+1} \otimes T$ since S^{n+1} is considered as S^n-module with the use of ε_j. Thus (1.2) implies that $\det \varepsilon_j a = \text{Norm}(S^{n+1} \otimes T/S^{n+1}; \varepsilon_j a) = \varepsilon_j \text{Norm}(S^{n+1}/S^n; a)$, i.e., $\tau\varepsilon_j = \varepsilon_j\tau$.

The proof of $\tau v_j = v_j\tau$ follows as in [2, p. 6] with the use of the transitivity of the Norm (Corollary 4).

REMARK. Theorem 2.6 of [2] considers also the homomorphism $\mu: S \otimes T \to T$ when $T \subseteq S$, and it is given by $\mu(j \otimes t) = jt$. Nevertheless, μ induces only a homomorphism for the *cohomology groups*: $H^*(S \otimes T/T) \to H^*(S/T)$ and the proof for homology group fails.

The relation between the transfer and restriction, namely, that $\tau^*\rho^* =$ dimension of S over R, is not true in this form. But:

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(2) This is defined for both $C_*(\ ,\ )$ and $C^*(\ ,\ )$, and the results are valid for both homology and cohomology groups.

THEOREM 5C. *$\tau^*\rho^*(a) = \Sigma e_j a^j$, where the idempotents e_j are given in (1.1). In particular, if all p-ranks of S over R are n then $\tau^*\rho^*(a) = a^n$.*

This is an immediate consequence of (c) of proposition 1.

An important application of the Norm is the following:

THEOREM 6. *If S is a f.g. projective R-algebra; and if the p-rank of S is m for every prime ideal p of R then the elements of $H(S/R)$(²) are of order dividing m. In the general case if m is the maximal p-rank of S, then the elements of $H(S/R)$ are of order dividing m!.*

Proof. As in the proof of [2] Theorem 2.10, we consider the homotopy $v: S^{n+1} \to S^n$, which is defined as $v = v_{n+1} = \varepsilon_{n+1}^{-1}$ Norm $(S^{n+1}/\varepsilon_{n+1}S^n, a)$. It follows from (2.4) that $\varepsilon_i v_n = v_{n+1}\varepsilon_i$, and therefore (when written additively):

$$\delta = \Delta v_n - v_{n+1}\Delta = (\Sigma(-1)^i\varepsilon_i)v_n - v_{n+1}\Sigma(-1)^i\varepsilon_i = (-1)^{n+1}v_{n+1}\varepsilon_{n+1}$$

where $\Delta: (S^n)^* \to (S^{n+1})^*$ is the derivation of $C^*(S/R)$. Proposition 1 yields that $v_{n+1}\varepsilon_{n+1}(a) = \Sigma e_i a^i$ and if the p-rank S is m for every m, $e_m = 1$ and all $e_j = 0$; hence $v_{n+1}\varepsilon_{n+1}(a) = a^m$ which proves the first part of the theorem since $v_{n+1}\varepsilon_{n+1}$ is homotopic with zero.

To prove the second part, we note that $\Sigma e_i a^i = \Delta b$ for some b since $v_{n+1}\varepsilon_{n+1}(a) = \Sigma e_i a^i$ and $v_{n+1}\varepsilon_{n+1}$ is homotopic with zero. Now $e_i\varepsilon_j a = \varepsilon_j e_i a$ for $e_i \in R$, and let $iv_i = m!$, thus set $w = \Sigma e_i v^{v_i}$. Clearly w has an inverse and

$$\Delta w = \Sigma e_i(\Delta b)^{\mu_i} = \Sigma e_i(\Sigma e_j a^j)^{\mu_i} = \Sigma e_i a^{m!} = a^{m!}.$$

The proof for homology groups follows the proof of [2] theorem 2.10 using the homotopy $\varepsilon = \varepsilon_{n+1}$, and first observing that $Nw = \Sigma e_i(Nb)^{\mu_i}$ since $e_i v_j(a) = v_j(e_i a)$ for idempotents e_i of the base ring R, and finally considering $S^n = e_j S^n \oplus (1 - e_j)S^n$ as modules over $e_j R \oplus (1 - e_j)R$.

3. **The inseparable case.** Let F be an inseparable field extension of a field C of characteristic p. The purpose of this section is to give an alternative proof for the following theorem of Berkson [3].

THEOREM 7. *$H^k(F/C) = 0$ for $k \geqq 3$.*

First we reduce the theorem to the case that $F = C(\xi)$ is generated by a single element ξ satisfying an equation $\xi^p - c = 0$. Indeed, if F is not of this form, then $F \supset I \supset C$ for some K which is also inseparable over C, and F is inseparable over K. It follows by [5] Theorem 4.3 that there is an exact sequence.

$$\cdots \to H^k(K/C) \to H^k(K/C) \to H^k(F/K) \to H^{k+1}(K/C) \to \cdots$$

and a simple induction process on the degree of the extensions will yield that $H^k(K/C) = H^k(F/K) = 0$ for $k \geqq 3$; hence, the exactness yields that $H^k(F/C) = 0$ for $k \geqq 3$.

In case $F = C(\xi)$, then the mapping $\mu : F \otimes F \to F$ given by $\mu(a \otimes b) = ab$ has a kernel N which is an $1 \otimes F = F$-algebra generated by the single element $\bar{\xi} = \xi \otimes 1 - 1 \otimes \xi$; furthermore $N^p = 0$ and $H^k(F/C) = 0$ for $k \geqq 3$ is a consequence from the following general result:

LEMMA 8. *Let S be an R-algebra of characteristic $p \neq 0$ for which the map $R \to S$ splits as an R-module map. Let μ: $S \otimes S \to S$ be the multiplication homomorphism i.e., $\mu(a \otimes b) = ab$, and let $N = \mathrm{Ker}\,\mu$. If $N^p = 0$ then $H^k(S/R) = 0$ for $k \geqq 3$.*

Proof. Consider the complex $C_1(S/R)$:

$$(S^2)^* \to (S^3)^* \to \cdots \to (S^k)^* \to \cdots$$

which is obtained from $C^*(S/R)$ by chopping its first term S. Let $\mu = \mu_1 : S^k \to S^{k-1}$ (k $\geqq 2$) be given by: $\mu_1(a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_n) = a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n$; that is, $\mu_1 = \mu \otimes 1 \otimes \cdots \otimes 1$.

First we show that μ_1 yields the following exact sequence:

$$(8.1) \qquad 1 \to (1 + \mathcal{N})^* \to C_1(S/R) \xrightarrow{\mu_1} C_1{}^*(S/R, \otimes S) \to 1$$

and where we denote by $C_1^*(S/R, \otimes S)$ is the complex composed of the groups $S^* \to (S \otimes S)^* \to (S \otimes S^2)^* \to \cdots \to (S \otimes S^{k-1})^* \to \cdots$ with the derivation is given by (written additively) $\Delta_1 = \varepsilon_2 - \varepsilon_3 + \cdots \pm \varepsilon_n$ and thus does not affect the first term; and $(1 + \mathcal{N})^* = \mathrm{Ker}\,\mu_1$ is the complex:

$$(8.2) \qquad (1 + N)^* \to (1 + N \otimes S)^* \to \cdots \to (1 + N \otimes S^{k-2})^* \to \cdots$$

Indeed, since $\mu_1\varepsilon_1 = \mu_1\varepsilon_2$ and $\mu_1\varepsilon_i = \varepsilon_{i-1}$ for $i \geqq 2$, it follows that

$$\mu_1\Delta = \mu_1(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \cdots) = (\varepsilon_2 - \varepsilon_3 + \cdots)\mu_1 = \Delta_1\mu_1$$

and hence μ_1 is a complex homomorphism, and from which we get the exact sequence (8.1).

It follows now by [2] theorem 2.9 that the homology groups of $C_1(S/R, \otimes S)$ are zero; hence, the exactness of (8.1) yields that

$$H^k[(1 + \mathcal{N})^*] \cong H^k[C_1(S/R)] = H^{k+1}(S/R).$$

The latter follows from the fact that the groups of $C_1(S/R)$ are those of $C^*(S/R)$ but shifted by 1.

To compute $H^k[(1 + \mathcal{N})^*]$ we pass to an additive complex $\mathcal{N}$ by considering the following two maps(3):

(3) This method is the same as in [1, p, 103], but there it was misused (as pointed out by Zelinsky-Rosenberg) since generally E and L have not the standard properties of the exponential and logarithm, even if every element is nilpotent of exponent p. Nevertheless, these properties hold if $N^p = 0$ and when this assumption is not applicable, a different method for computation should be applied.

For every $n \in N \otimes S^{k-2}$ ($k \geqq 2$), we define:

$$E(n) = 1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{p-1}}{(p-1)!}$$

we define

$$L(1 + n) = n - \frac{n^2}{2} + \cdots + (-1)^p \frac{n^{p-1}}{p-1}$$

and we prove first that E maps the additive group of $N_k = N \otimes S^{k-2}$ onto the multiplicative group $(1 + N_k)^*$, and its inverse is the map L.

Consider the ring of power series in two indeterminates t, s with coefficients in the field Q of all rational numbers, and let

$$\mathrm{Exp}\, t = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \text{ and } \mathrm{Log}(1 + t) = \sum_{\nu=0}^{\infty} (-1)^{\nu-1} \frac{t^\nu}{\nu}$$

then we have

$$\mathrm{Exp}\, t \cdot \mathrm{Exp}\, s = \mathrm{Exp}(t + s) \text{ and } \mathrm{Log}(1 + t) + \mathrm{Log}(1 + s) = \mathrm{Log}([1 + t)(1 + s)]$$

and $\mathrm{Exp}\,\mathrm{Log}(1 + t) = 1 + t$, $\mathrm{Log}\,\mathrm{Exp}\, t = t$.

Now, $E(t) = \mathrm{Exp}\, t - R(t)$, $L(1 + t) = \mathrm{Log}(1 + t) - S(t)$, where $R(t)$ and $S(t)$ are power series in t containning powers of t with exponents $\geqq p$. Thus, (8.2a) $E(t)E(s) = [\mathrm{Exp}\, t - R(t)]\ [\mathrm{Exp}\, s - R(s)] = \mathrm{Exp}(t+s) + U(t,s) = E(t + s) + V(t,s)$ and $V(t,s) = R[t + s] - R(t)\,\mathrm{Exp}\, s - \mathrm{Exp}\, t \cdot R(s) + R(t)R(s) \equiv 0 \pmod{[t^p, s^p, (t + s)^p]}$. Hence identifying coefficients of both sides of (8.1a) we get that

$$E(t)E(s) = E(t + s) + \sum_{i+j \geqq p} \lambda_{ij} t^i s^j$$

and the coefficient λ_{ij} are rational numbers. The coefficient on the left side do not have denominators prime to p, hence so are the λ ; consequently the last equality will hold also in any algebra over a field of characteristic p. In particular, in our case, where $N_k^p = 0$, we obtain $E(n)E(m) = E(n + m)$ as $\sum_{i+j \geqq p} \lambda_{ij} n^i m^j \in N_k^p$ for $n, m \in N_k$.

Using the other properties of Exp, Log quoted above, one obtains the proof of the facts that $L\colon (1 + N_k)^* \to N_k$ and it is the inverse of E.

By applying E on each of the component of the complex $(1 + \mathcal{N})^*$ given in (8.2) we obtain that $(1 + \mathcal{N})^*$ is isomorphic to the complex $\mathcal{N}$.

(8.3) $$N \to N \otimes S \to \cdots \to N \otimes S^k \to \cdots$$

whose groups are the additive groups $N_k = N \otimes S^{k-2}$ and with a derivation $\delta = L\Delta^* E$, where $\Delta^* = \pi \varepsilon_i^{+1}$ is the derivation of the multiplicative groups $(1 + N_k)^*$.

Our next aim is to show that:

$$\delta = \Delta^{+} + \lambda = \varepsilon_1 - \varepsilon_2 + \varepsilon_3 \cdots + (-1)^k \varepsilon_k + \lambda \tag{8.4}$$

and where

$$\lambda(n) = \sum_{\nu=1}^{p-1} (-1)^{\nu} \frac{1}{\nu!(p-\nu)!} (\varepsilon_1 n)^{\nu} (\varepsilon_2 n)^{p-\nu} \left(= \frac{1}{p!} [[(\varepsilon_1 - \varepsilon_2)]^p - (\varepsilon_1 n)^p) + (\varepsilon_2 n)^p] \right)$$

and the last equality is to be considered only in a formal way, where the binomial expansion and division by $p!$ should be applied first.

Indeed, if $q \in 1 + N \otimes S^{k-2}$, then $\varepsilon_j q$ for $j \geqq 3$ belongs to $1 + N \otimes S^{k-1}$ since $\mu_1 \varepsilon_j = \varepsilon_j \mu_1$; but $\varepsilon_1 q$ will not necessarily belong to $1 + N \otimes S^{k-1}$. Nevertheless, the relation $\mu_1 \varepsilon_1 = \mu_1 \varepsilon_2$ implies that $(\varepsilon_1 q)(\varepsilon_2 q)^{-1} \in \operatorname{Ker} \mu_1 = 1 + N \otimes S^{k-1}$; hence $\varepsilon_1 \varepsilon_2{}^{-1}: 1 + N \otimes S^{k-2} \to 1 + N \otimes S^{k-1}$. Thus for every $m \in N \otimes S^{k-2}$ we have $Em \in 1 + N \otimes S^{k-2}$ and for $j \geqq 3$ $L \in \varepsilon_j Em = LE\varepsilon_j m = \varepsilon_j m$.

Consequently

$$\begin{aligned} \text{(8.4a)} \quad \delta m = L\Delta^* E(m) &= (L\varepsilon_1 \varepsilon_2^{-1} E)(m) + \varepsilon_3(m) - + \cdots \pm \varepsilon_k(m) = \\ &= (L\varepsilon_1 \varepsilon_2^{-1} E)(m) - (\varepsilon_1 - \varepsilon_2) m + \Delta(m) = \lambda(m) + \Delta(m) \end{aligned}$$

where $\lambda = L\varepsilon_1 \varepsilon_2^{-1} E - (\varepsilon_1 - \varepsilon_2)$. Finally we show that λ has the form given in (8.4).

To this end we consider two commutative indeterminates t, s over a prime field $GF(p)$ of p elements. Let:

$$L(Et \cdot Es) = Q[t, y] + x^p M_t + y^p M_s \tag{8.4b}$$

where Q contains all monomials of degree at most $p - 1$ in t and the same in s and M_t, M_s contain the other terms. Let $D = (d/dt)$ be the formal derivation with respect to t, then note that $D_t t^p = 0$ in a ring of characteristic p. Apply D on both sides of (8.4b) and obtain:

$$\begin{aligned} DQ[t,s] + t^p DM_t + s^p DM_s &= D\left[L(EtEs)\right] = D \left[\sum_1^{p-1} (-1)^{\nu-1} \frac{(Et \cdot Es - 1)^{\nu}}{\nu}\right] \\ &= \sum_1^{p-1} (-1)^{\nu-1} (Et \cdot Es - 1)^{\nu-1} D[Et \cdot Es - 1]. \end{aligned}$$

Now, $D(Et) = Et - (t^{p-1}/(p-1)!)$. Hence we obtain the following relation modulo (t^{p-1}, s^p):

$$\begin{aligned} DQ \equiv \sum_1^{p-1} (-1)^{\nu-1} (Et \cdot Es - 1)^{\nu-1} Et \cdot Es &= \sum_1^{p-1} (-1)^{\nu-1} (EtEs - 1)^{\nu-1} \\ &= 1 - (1 - EtEs)^{p-1}. \end{aligned}$$

Next we use the relation, $(u+v)^{p-1} = \dfrac{(u+v)^p}{u+v} = \dfrac{u^p+v^p}{u+v} = \sum_{\nu=0}^{p-1} u^\nu v^{p-1-\nu}$

(which holds in every ring of characteristic p), and the properties of E, to obtain that $\mathrm{mod}(t^{p-1}, s^p)$

$$DQ = 1-(1-EtEs)^{p-1} \equiv 1 - \sum_{\nu=0}^{p-1}(Et\cdot Es)^\nu \equiv 1 - \sum_{\nu=0}^{p-1} E(\nu t)\cdot E(\nu s)$$

$$1-1+\sum_{\nu=1}^{p-1}\ \sum_{k,h=0}^{p-1} \frac{(\nu t)^k}{k!}\frac{(\nu s)^h}{h!}\ \sum \frac{t^k s^h}{k!h!}\sum_{\nu=1}^{p-1}\nu^{k+h}.$$

But $\sum_1^{p-1} v^{k+h} \equiv (\mathrm{mod}\, p)$ only if $k+h\equiv 0\ (p-1)$. Thus, the last sum will contain possible non zero terms when $k+h=0,\ p-1,\ 2p-1$. The first is equal to 1 and the last is a term containing monomial of degree $p-1$ in x. Hence

$$DQ \equiv 1 + \sum_{k+h=p-1} \frac{t^k s^h}{k!h!} \mathrm{mod}(t^{p-1}, s^p).$$

We actually have here an equality, since DQ contains only monomials of degree $\leqq p-2$ in t and of degree $\leqq p-1$ in s. Hence, by integrating, it follows that

$$Q = t + \sum_{k+h=p-1}\frac{t^{k+1}s^k}{(k+1)!h!} + f(s)$$

and where f does not contain monomials in t. It follows immediately from the definition of Q in (8.4b) that it must the symmetric in t and s, so we must have $f(s)=a$

$$Q[t,s] = t+s+\sum_{\nu=1}^{p-1}\frac{t^\nu s^{p-\nu}}{\nu!(p-\nu)!}\ \left(=t+s+\frac{1}{p!}\Big[(t+s)^p - t^p - s^p\Big]\right).$$

the last form of Q together with (8.4b) can be applied to our case, in the following way:

set $t=\varepsilon_1 m,\ s=-\varepsilon_2 m$. Since $(\varepsilon_1 m)^p=(\varepsilon_2 m)^p=0$, we have $[E(\varepsilon_2 m)]^{-1}=E(-\varepsilon_2 m)$, and be (8.4b)

$$\begin{aligned}(L\varepsilon_1\varepsilon_2^{-1}E)m &= L[(\varepsilon_1 E_m)(\varepsilon_2 E_m)^{-1}] = L[\ (E\varepsilon_1 m)E(-\varepsilon_2 m)]\\ &= Q[\varepsilon_1 m, -\varepsilon_2 m]\\ &= \varepsilon_1 m - \varepsilon_2 m + \sum_{\nu=1}^{p-1}\frac{(\varepsilon_1 m)(-\varepsilon_2 m)^{p-\nu}}{\nu!(p-\nu)!}\end{aligned}$$

and this completes the proof of (8.4).

Finally, we return to the computation of the groups of the complex given in (8.3):

Note that it follows by the formula for λ, that for every $n \in N$ and $a \in S^{k-2}$, $\lambda(n \otimes a) = \lambda(n)a^p \in N \otimes S \otimes 1 \otimes \cdots \otimes 1$, since $\varepsilon_i(n \otimes a) = \varepsilon_i n \otimes a$ for $i = 1, 2$. This being true for all generators of $N \otimes S^{k-2}$ implies that the mapping λ maps $N \otimes S^{k-2}$ into the sub-ring $N \otimes S \otimes 1 \otimes \cdots \otimes 1$ of $N \otimes S^{k-1}$. Consider now the map $\eta: S - R$ which splits the R-map: $R \to S$ and extend it to a contraction map η: $S \to S^{k-1}$ by setting $\eta(a_1 \otimes \cdots \otimes a_k) = (-1)^{k-1} a_1 \otimes \cdots \otimes a_{k-1}\ \eta(a_k)$. Then clearly $\mathscr{N}$ maps $N \otimes S^{k-1} \to N \otimes S^{k-2}$ for $k \geqq 2$; it does not affect the first two factors.

One readily verifies that $\eta\varepsilon_i = \varepsilon_i\eta$ for $i = 1, 2, \cdots, k-1$ and $\eta\,\varepsilon_k = (-1)^{k-1}$. Hence, $\eta\,\Delta - \Delta\,\eta = 1$. Moreover, $\mu\lambda = \pm\,\lambda$ and $\lambda\,\eta(m) = \mp\,\lambda(m)$ for every $m \in N \otimes S \otimes 1 \otimes \cdots \otimes 1$. From which we conclude what if $b \in N \otimes S^{k-2}$ is a cocycle, i.e., $\delta b = (\Delta + \lambda)b = 0$, then

$$b + (\eta\lambda - \lambda\eta)b = [\eta(\Delta + \lambda) - (\Delta + \lambda)\eta]b = -(\Delta + \lambda)\eta(b).$$

Thus b is cohomologous to $b_0 = (\eta\,\lambda - \lambda\,\eta)b \in N \otimes S \otimes 1 \otimes \cdots \otimes 1$. So if $b \in N \otimes S^{k-2}$ and $k \geqq 3$, we can choose in the class of n of $H^{k-2}(\eta)$ a representative b_0 in $N \otimes S \otimes 1 \otimes \cdots \otimes 1$. But then again $b_0 v - (\eta\,\lambda - \lambda\eta)b_0$, and for these elements $(\eta\lambda - \lambda\,\eta)b_0 = \pm\,\lambda b_0 - \lambda b_0 = 0$, which proves that $b \sim b_0 \sim 0$, i.e., $H^{k-2}(\mathscr{N}) = 0$ for $k \geqq 4$.

Hence $0 = H^{k-2}(\mathscr{N}) = H^{k-2}((1 + \mathscr{N})^*) \cong H^{k-1}(S/R)$, which completes the proof of lemma 8, and theorem 7.

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